SPHERICAL ASTRONOMY TEST
Sep 27, 2017

1. Here's a geometric problem for warm-up: given that we have just witnessed an awesome solar eclipse, compute what is the longest possible duration of an eclipse. The radius of the Sun is $696,000\,\text{km}$, the radius of Earth is $6378.1\,\text{km}$, the semi-major axis of Earth's orbit is 149.6 million km, and orbital eccentricity is 0.0167. The radius of the Moon is $1738.1\,\text{km}$, the semi-major axis of its orbit around Earth is $384,400\,\text{km}$, and orbital eccentricity is 0.0549. The average length of 1 month is 29.5 days. If you had forgotten an equation for the ellipse, here it is:

$$r(\upsilon) = \frac{a(1-e^2)}{1+e\cos\upsilon},$$

where υ is the angle measured from perihelion, i.e. Earth's closest approach to the Sun.

Answer. The longest eclipse will occur when the Sun is the farthest from Earth and the Moon is the closest to Earth. From the ellipse equation it follows that these two distances are:

$$a_{\text{min}} = \frac{a(1-e^2)}{1+e} = a(1-e), \quad a_{\text{max}} = \frac{a(1-e^2)}{1-e} = a(1+e).$$

The angular size of the lunar shadow as seen from the surface of Earth can then be written as:

$$\theta_{\text{shadow}} = \frac{R_M}{a_M(1-e_M) - R_E} = 4.866 \times 10^{-3} rd \equiv 0.279^\circ.$$

Equivalently, the angular size of the Sun as seen from the surface of Earth can be written as:

$$\theta_\odot = \frac{R_\odot}{a_E(1+e_E) - R_E} = 4.573 \times 10^{-3} rd \equiv 0.262^\circ.$$

The displacement in time is predominantly governed by Moon's orbital revolution about Earth. Thus:

$$\omega = \frac{2\pi}{t_M} = \frac{\Delta\theta}{\Delta t},$$

where $\Delta\theta$ is the angular difference between the shadow size and the Sun size, i.e. $\Delta\theta = 2(\theta_{\text{shadow}} - \theta_\odot)$. Solving for Δt and plugging in the numbers yields:

$$\Delta t = \frac{t_M}{2\pi}\Delta\theta = 3.96\,\text{min}.$$

2. This class' favorite star is undoubtedly Deneb (right, Kasey?). Its equatorial coordinates are $\alpha = 20^{\rm h}\,41^{\rm m}\,26^{\rm s}$ and $\delta = 45^\circ\,16'\,49''$. We are observing tonight (Sep 27, 2017) from Villanova ($\varphi = 40^\circ\,02'\,14''$ N, $\lambda = 75^\circ\,20'\,57''$ W).

   a) At what time(s) will Deneb be seen at 60° above the horizon?
   b) At what altitude will Deneb be seen when it crosses the E-W meridian?
   c) On what date in 2018 will Deneb culminate at 9:30pm?

   Neglect any effects of refraction or precession.

   Answer. Start off by looking up the sidereal time at midnight UT from the Almanac:

$$S(0^{\rm hr}\ \ {\rm UT}) = 0^{\rm h}\,24^{\rm m}\,51^{\rm s}.$$

For part (a), solve the altitude equation for the hour angle:

$$\cos H = \frac{\sin h - \sin\varphi\sin\delta}{\cos\varphi\cos\delta}$$

This equation has two solutions, $\pm 2^{\rm h}\,42^{\rm m}\,28^{\rm s}$. We need to compute times for both. We use the equation for sidereal time for that:

$$S_i = H_i - \alpha = S(0^{\rm hr}\ \ {\rm UT}) - \lambda + \frac{1}{\gamma}(t_i + TZ - DST), \quad i = 1, 2.$$

Here λ needs to be given in hours, i.e. $\lambda = 5^{\rm h}\,01^{\rm m}\,23.8^{\rm s}$, and $TZ - DST = 4^{\rm hr}$ because we are still in daylight savings mode. The two solutions are:

$$t_1 = \gamma(-H + \alpha - S(0^{\rm hr}\ \ {\rm UT}) + \lambda) - 4^{\rm hr} = 18^{\rm h}\,31^{\rm m}\,49^{\rm s},$$
$$t_2 = \gamma(H + \alpha - S(0^{\rm hr}\ \ {\rm UT}) + \lambda) - 4^{\rm hr} = 23^{\rm h}\,55^{\rm m}\,51^{\rm s}.$$

Note that you would get the same answer for t_2 from $t_2 = t_1 + 2\gamma|H|$.

For part (b), no computation is necessary: as $\varphi + \delta < 90^\circ$, the object will never cross the E-W meridian. To see it algebraically, crossing the E-W meridian implies that $A = \pm 90^\circ$. As this is symmetrical w.r.t. conjunction, both values would yield the same altitude, so we would need to solve for either value of the azimuth:

$$\cos A = 0 = \sin\delta - \sin\varphi\sin h \quad \Rightarrow \quad \sin h = \frac{\sin\delta}{\sin\varphi} > 1.$$

Finally, for part (c), the condition for culmination is $S = \alpha$. From here we can solve for $S(0^{\rm hr}\ \ {\rm UT})$:

$$S(0^{\rm hr}\ \ {\rm UT}) = \alpha + \lambda - \frac{1}{\gamma}(21.5^{\rm hr} + 4^{\rm hr}) = 00^{\rm h}\,08^{\rm m}\,38^{\rm s},$$

which, from the Almanac, corresponds to Sep 23, 2018.

3. Redo problem (2), but this time take refraction and precession corrections into account. Assume that the equatorial coordinates of Deneb are given in the epoch 2000.0.

   Answer. First, let us precess equatorial coordinates from epoch 2000.0 to present day; for $\theta = 50''/\text{year}$, $\Delta\alpha = \alpha_{2017.8} - \alpha_{2000.0}$ and $\Delta\delta = \delta_{2017.8} - \delta_{2000.0}$:

$$\Delta\alpha = (2017.8 - 2000.0)\theta(\cos\varepsilon + \sin\varepsilon \sin\alpha \tan\delta),$$

$$\Delta\delta = (2017.8 - 2000.0)\theta \sin\varepsilon \cos\alpha;$$

$$\alpha_{2017.8} = 20^{\text{h}}\, 42^{\text{m}}\, 02^{\text{s}}, \delta_{2017.8} = 45^\circ\, 20'\, 39''.$$

   We now have the updated α and δ to work with, and our account for precession is done.

   For part (a), we need to compute the actual altitude h_0 from the observed altitude $h = 90^\circ - \zeta = 60^\circ$. As:

$$R = z - \zeta = k \tan\zeta \quad \Rightarrow \quad z = \zeta + k\tan\zeta = 30^\circ\, 00'\, 34'',$$

   the actual altitude is $h_0 = 90^\circ - z = 59^\circ\, 59'\, 26''$. From now on everything else is the same as before. We get:

$$H = 2^{\text{h}}\, 42^{\text{m}}\, 33^{\text{s}}, \quad t_1 = 18^{\text{h}}\, 32^{\text{m}}\, 20^{\text{s}}, \quad t_2 = 23^{\text{h}}\, 56^{\text{m}}\, 32^{\text{s}}.$$

   There is no change for part (b); and for part (c), we get:

$$S(0^{\text{hr}}\ \text{UT}) = 00^{\text{h}}\, 09^{\text{m}}\, 14^{\text{s}}.$$

4. Two watchtowers at known geographic coordinates (φ_A, λ_A) and (φ_B, λ_B) detect a distress radio signal from a ship at unknown geographic coordinates (φ_S, λ_S). Each watchtower measures the angle w.r.t. geographic north at which the signal is received; denote these angles with δ_A and δ_B for watchtower A and B, respectively. What are the unknown coordinates of the ship? Fun fact: this method is known as *triangulation*.

Answer. There are many ways to solve this problem. We present one solution here. Let us denote the north pole with P.

ΔABP: we can compute side $p = AB$ from the law of cosines, and angles at A and B from the law of sines:

$$\cos p = \sin\varphi_A \sin\varphi_B + \cos\varphi_A \cos\varphi_B \cos\Delta\lambda,$$

$$\sin A = \frac{\sin \Delta\lambda \cos \varphi_B}{\sin p}, \quad \sin B = \frac{\sin \Delta\lambda \cos \varphi_A}{\sin p}.$$

ΔASP, ΔSBP: let us denote the angle at A with α, side AS with c_1 and angle at P with $\Delta\lambda_1$. Analogously, let us denote the angle at B with β, side SB as c_2 and angle at P with $\Delta\lambda_2$. Angles α and β are given by the problem.

ΔABS: let us denote the angle at A with α' and angle at B with β'. As angles A, B, α, β are known, angles α' and β' are known as well:

$$\alpha' = A - \alpha, \quad \beta' = B - \beta.$$

Let us now use the law of sines for ΔASP, ΔSBP and ΔABS:

$$\frac{\sin \Delta\lambda_1}{\sin c_1} = \frac{\sin \alpha}{\cos \varphi_S}, \quad \frac{\sin \beta'}{\sin c_1} = \frac{\sin \alpha'}{\sin c_2}, \quad \frac{\sin \Delta\lambda_2}{\sin c_2} = \frac{\sin \beta}{\cos \varphi_S}.$$

This system can be solved for the ratio:

$$\frac{\sin \Delta\lambda_2}{\sin \Delta\lambda_1} = \frac{\sin \beta}{\sin \alpha} \frac{\sin \alpha'}{\sin \beta'}.$$

Noting that $\Delta\lambda = \Delta\lambda_1 + \Delta\lambda_2$, the above expression can be solved for either $\Delta\lambda_1$ or $\Delta\lambda_2$, which gives us the unknown longitude of the ship since $\Delta\lambda_1 = \lambda_S - \lambda_A$ and $\Delta\lambda_2 = \lambda_B - \lambda_S$.

Finally, to obtain the unknown latitude, we use side p and angles α' and β' in ΔABS to get c_1 or c_2:

$$\cos p \cos \beta' = \sin p \cot c_2 - \sin \beta' \cot \alpha'.$$

Once we have c_2, we obtain $varphi_S$ from the law of sines for ΔBPS:

$$\cos \varphi_S = \frac{\sin \beta \sin c_2}{\sin \Delta\lambda_2}.$$

Test time: 150 minutes. The best of luck, ladies and gents, make me proud!