POPULATION MODELS

Due date: 2/15/2022, 4pm

The equations of population theory, as we discussed in class, stem from rate equations for chemical reactions. These are systems of non-linear first-order ordinary differential equations, so we already know how to solve them. The new part here will be to explore the phase spaces thoroughly and meaningfully.

1. **Logistic equation.** After the initial hype (and failure) of the basic (exponential) population growth model, we need to incorporate resource limitations into the model. These are described with the logistic equation (the Verhulst model):

   $$\dot{P} = kP\left(1 - \frac{P}{N}\right),$$

   where P is population size, k is the growth rate, and N is the size constraint. Look up US (or any other) population as a function of time, and compare the fit of the logistic model to the fit of the exponential growth model. What are suitable values of k and N? What do they imply?

2. **Predator–prey.** We have shown in class that the number of independent parameters for the Lotka-Volterra model can be reduced to 1; draw and carefully analyze the phase diagram. Determine the population equilibrium (the point in the phase diagram where the population levels of foxes and rabbits are not changing) and points of stability. Study the behavior around the fixed points. What happens if you add wolves into the system?

3. **The epidemic equation.** Divide the population into three groups: (1) healthy, (2) sick, and (3) immune. The disease spreads by contact between the sick and the healthy with some constant probability. The sick become immune (either healthy or dead) by another constant probability. Form the model and determine the minimum number of independent parameters to describe it. During the epidemic, we are interested in: the time of the peak in the number of sick people; the overall number of sick people; and the expected maximum number of sick people at any one time. Determine these quantities for some realistic values of parameter(s). How does innoculation change these results? Once you have the basic model going, try to introduce disease progression, incubation periods, immunity expiration, etc.

   Hint: to integrate differential equations that feature a time delay, i.e. $y'(x) = f(x, y(x), y(x - \tau))$, look up delay differential equations.